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Complexified Vaidya's Metric

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It is shown that the complexification of Vaidya's metric results in a metric that bears the same relation to Vaidya's metric as does the Kerr metric to the Schwarzschild metric. However, the energy–momentum tensor consists of two parts: (1) radiative and (2) nonradiative. The nonradiative part also corresponds to a trace-free field.

1. INTRODUCTION

In a recent paper Carmeli and Kaye (1975) obtain a metric which bears the same relation to the Kerr (1969) metric as does the Vaidya (1951) metric to the Schwarzschild metric. Their solution is algebraically special of Petrov type II with a twisting, sheer-free, null congruence identical to that of the Kerr metric. However, the energy–momentum tensor contains an extra term in addition to the radiative term. The extra term describes a nonradiative field. In generalizing Vaidya's metric Hughstone (1971) obtained criteria under which one can construct a metric compatible with the energy–momentum tensor of a null radiation field from an algebraically special vacuum metric. It turns out that the Kerr metric does not satisfy these criteria, and therefore it is not possible to use it for constructing a null radiation field corresponding to the Kerr metric.

Newman and Janis (1965) presented a "curious" derivation of the Kerr metric by performing a complex coordinate transformation on the Schwarzschild metric. Later Talbot (1969) showed why their complex coordinate trick works for solutions of the vacuum field equations. In the present paper we have studied the complex coordinate transformation for the radiative metric. It is found that the complex coordinate transformation, which leads the Schwarzschild metric to the Kerr metric, works equally well on Vaidya's metric to yield the metric obtained by Carmeli

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and Kaye. In Section 2 we present the algebraically special metric using the notation of Robinson and Robinson (1969). Section 3 contains the presentation of Vaidya's metric following the prescription of Hughstone. Section 4 deals with the coordinate transformation which leads to Carmeli's metric, and the energy–momentum tensor of the derived metric is given.

2. ROBINSON'S SOLUTION

Robinson and Robinson (1969) have shown that if a space–time with signature $(+,+,+,-)$ admits a null vector k^{μ} tangent to a nonshearing, diverging congruence of affinely parametrized null geodesic curves, then coordinates

$$x^{\mu}=(\xi,\bar{\xi},\sigma,\rho) \tag{2.1}$$

may be chosen such that in a vacuum the main gravitational field equations

$$k_{[\mu}R_{\nu][\sigma}k_{\gamma]}=0 \tag{2.2}$$

have the solutions

$$ds^2=2P^2\,d\xi\,d\bar{\xi}+2\,d\Sigma\left(d\rho+Z\,d\xi+\bar{Z}\,d\bar{\xi}+S\,d\Sigma\right) \tag{2.3a}$$

$$d\Sigma=a\left(b\,d\xi+\bar{b}\,d\bar{\xi}+d\sigma\right)=k_{\mu}\,dx^{\mu} \tag{2.3b}$$

$$S=\rho u_3-1/4(K+\bar{K})+(m\rho+\Omega M)/(\rho^2+\Omega^2) \tag{2.3c}$$

$$K=2\exp(-2u)L_2 \tag{2.3d}$$

$$L=\Lambda-u_1 \tag{2.3e}$$

$$M=\left[(K+\bar{K})/2\right]\Omega+(1/2)\exp(-2u)\left[(\Omega_1+\Lambda\Omega)_2+(\Omega_1+\Omega\Lambda)\bar{\Lambda}\right.$$
$$\left.+\left(\Omega_2+\Omega\bar{\Lambda}\right)_1+\Lambda(\Omega_2+\bar{\Lambda}\Omega)\right] \tag{2.3f}$$

$$\Omega=(1/2)ia\exp(-2u)\left(\bar{b}_1-b_2\right) \tag{2.3g}$$

$$P^{-1}=\exp(-u)z \tag{2.3h}$$

$$z=\theta+iw=\frac{\rho}{\rho^2+\Omega^2}-i\frac{\Omega}{\rho^2+\Omega^2} \tag{2.3i}$$

$$\Lambda=a^{-1}a_1-ab_3 \tag{2.3j}$$

$$Z=\rho\Lambda-(\Omega_1+\Omega\Lambda)i \tag{2.3k}$$

where a, b, u, m are functions of $\xi, \bar{\xi}, \sigma$ and w, θ are twist and expansion, subject only to

$$a \neq 0 \tag{2.4}$$

We use the notation

$$df = f_1 d\xi + f_2 d\bar{\xi} + f_3 d\Sigma \tag{2.5}$$

for any function

$$f(\xi, \bar{\xi}, \sigma)$$

Robinson and Robinson (1969) note that in a vacuum the remaining subsidiary field equations are equivalent to the vanishing of a form dC:

$$dC = \left[\rho^{-3}(m - iM)\right]_{/1} d\xi + \left[(m + iM)\rho^{-3}\right]_{/2} d\bar{\xi} + \left\{\left[(m + iM)\rho^{-3}\right]_{/3} + \rho^{-4}\exp(-4u)I\right\}\rho\, d\Sigma = 0 \tag{2.6}$$

where we define

$$I = J_{22} + 2\bar{L}J_2 \tag{2.7}$$

$$J = L_1 + L^2 \tag{2.8}$$

and use the notation

$$df = f_{/1} d\xi + f_{/2} d\bar{\xi} + f_{/3}\rho\, d\Sigma + f_{/4} dW \tag{2.9}$$

$$dW = \Lambda\, d\xi + \bar{\Lambda}\, d\bar{\xi} + u_3\, d\Sigma + \rho^{-1} d\rho \tag{2.10}$$

for any function $f(\xi, \bar{\xi}, \sigma, \rho)$.

3. VAIDYA'S METRIC

The metric given by (2.3) may be expressed as

$$ds^2 = e_{a\mu} e_{b\nu} g^{ab}\, dx^\mu\, dx^\nu \tag{3.1}$$

where

$$e_{a\mu} = \left(\bar{m}_\mu, m_\mu, n_\mu, k_\mu\right)$$

are null tetrads satisfying

$$e^{\mu}{}_{a}e^{b}{}_{\mu}=\delta^{b}_{a}, \qquad e^{\mu}{}_{a}e^{a}{}_{\nu}=\delta^{\mu}{}_{\nu} \tag{3.2}$$

$$e^{\mu}{}_{1}=\bar{m}^{\mu}=P^{-1}\delta^{\mu}{}_{1}-bP^{-1}\delta^{\mu}{}_{3}-P^{-1}Z\delta^{\mu}{}_{4} \tag{3.3a}$$

$$e^{\mu}{}_{2}=m^{\mu}=\bar{P}^{-1}\delta^{\mu}{}_{2}-\bar{P}^{-1}\bar{b}\delta^{\mu}{}_{3}-\bar{Z}\bar{P}^{-1}\delta^{\mu}{}_{4} \tag{3.3b}$$

$$e^{\mu}{}_{3}=n^{\mu}=a^{-1}\delta^{\mu}{}_{3}-S\delta^{\mu}{}_{4} \tag{3.3c}$$

$$e^{\mu}{}_{4}=k^{\mu}=\delta^{\mu}{}_{4} \tag{3.3d}$$

and

$$g_{ab}=g^{ab}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=e_a\cdot e_b=e^a\cdot e^b \tag{3.4}$$

For $a=1$, $b=0$, $\exp(u)=2^{1/2}/(1+\xi\bar{\xi})$, $m=r_0=\text{const}$, the tetrads corresponding to the Schwarzschild metric in null coordinates take the form

$$e^{\mu}{}_{1}=\bar{m}^{\mu}=\left[2^{1/2}\exp(-u)/2\rho\right]\delta^{\mu}{}_{1} \tag{3.5a}$$

$$e^{\mu}{}_{2}=m^{\mu}=\left[2^{1/2}\exp(-u)/2\rho\right]\delta^{\mu}{}_{2} \tag{3.5b}$$

$$e^{\mu}{}_{3}=\eta^{\mu}=\delta^{\mu}{}_{3}+(1/2-r_0/\rho)\delta^{\mu}{}_{4} \tag{3.5c}$$

$$e^{\mu}{}_{4}=k^{\mu}=\delta^{\mu}{}_{4} \tag{3.5d}$$

To obtain Vaidya's metric from the Schwarzschild metric, we follow Hughstone's prescription:

$$H=h(\sigma)/\rho^3 \tag{3.6}$$

A substitution for $r_0\to m(\sigma)=r_0+h(\sigma)$ in (3.5) leads immediately to the tetrads corresponding to Vaidya's metric.

The energy–momentum tensor is given by

$$T_{\mu\nu}=\left[+2\dot{m}(\sigma)k_\mu k_\nu\right]/\rho^2 \tag{3.7}$$

$$k_\mu=\delta^{3}{}_{\mu} \tag{3.8}$$

Vaidya has shown that (3.7) represents the flow of radiation along the radial coordinate.

4. COMPLEXIFICATION OF VAIDYA'S METRIC

To complexify Vaidya's metric we allow ρ and σ take complex values,

$$\rho' = \rho - iT \tag{4.1}$$

$$\sigma' = \sigma - iR \tag{4.2}$$

where ρ' and σ' are real, and R and T are real functions of ξ and $\bar{\xi}$.

Such a complexification then demands that the function $m(\sigma)$ in $e^{\mu}{}_3 = \eta^{\mu}$ should be replaced by $m(\sigma')$. The new tetrads in terms of the new coordinate system yield a metric which is algebraically special with a vector k^{μ} tangent to the congruence of null, shear-free geodesics. The Ricci tensor is given by

$$R_{\mu\nu} = 2R_{a3}e^{a}_{(\mu}k_{\nu)} \qquad a \neq 4 \tag{4.3}$$

The following substitutions consistent with (4.1) and (4.2),

$$\begin{aligned} z = 1/\rho \to z' = 1/|\rho' + iT|; &\qquad \Omega = 0 \to \Omega' = T \\ b = 0 \to b' = -i\partial R/\partial\xi & \\ \exp(u) \to \exp(u) & \\ M = 0 \to M = 0 & \end{aligned} \tag{4.4}$$

keep the relations obtained from the solutions of the main equations (2.2) unchanged in equations.

Introducing the new values of b and Ω in (2.3f) and (2.3g) we obtain

$$\frac{\partial^2 T}{\partial\xi\,\partial\bar{\xi}} + \exp(2u)T = 0 \tag{4.5}$$

$$R = T + A, \qquad A = \text{const} \tag{4.6}$$

Using $A = 0$, R and T are given by

$$R = T = k\frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}} \qquad k = \text{const} \tag{4.7}$$

The metric (2.3) now takes the form

$$ds^2 = 2\exp(2u)(\rho'^2 + T^2)\,d\xi\,d\bar{\xi} + 2\,d\Sigma'\left\{ -i\frac{\partial T}{\partial \xi}\,d\xi + i\frac{\partial T}{\partial \xi}\,d\xi + i\frac{\partial T}{\partial x}\,d\bar{\xi} + \left[-1 + \frac{m(\sigma')T}{\rho'^2 + T^2} \right] d\Sigma' + d\rho' \right\} \tag{4.8}$$

where

$$d\Sigma' = -i\frac{\partial T}{\partial \xi}\,d\xi + i\frac{\partial T}{\partial \bar{\xi}}\,d\bar{\xi} + d\sigma' \tag{4.9}$$

The new Ricci tensor is given by

$$\begin{aligned} R_{\mu\nu} = & -2\dot{m}(z'\bar{z}')^2\rho'^2 k_\mu k_\nu \\ & + k_\mu k_\nu \left[2\frac{\partial T}{\partial \xi}\frac{\partial T}{\partial \bar{\xi}}\exp(-2u) \right]\left[\ddot{m}\rho'(z'\bar{z}')^2 - 2\dot{m}(z'\bar{z}')^3(\rho'^2 - T^2) \right] \\ & + i\frac{\partial T}{\partial \xi}\exp(-u)z'\bar{z}'^2\dot{m}(m_\mu k_\nu + m_\nu k_\mu) \\ & - i\frac{T}{\partial \bar{\xi}}\exp(-u)\bar{z}'z'^2\dot{m}(\bar{m}_\mu k_\nu + \bar{m}_\nu k_\mu) \end{aligned} \tag{4.10}$$

where

$$\dot{m} = \frac{\partial m}{\partial \sigma'}$$

Now, making the following coordinate transformation,

$$\begin{aligned} &\xi \to \theta \qquad \xi = \tan\tfrac{1}{2}\theta\exp(i\phi) \\ &\bar{\xi} \to \phi \\ &\sigma' \to \sigma' \\ &\rho' \to \rho' \end{aligned} \tag{4.11}$$

we obtain from (4.8) the metric obtained by Carmeli and Kaye (1975),

$$\begin{aligned} ds^2 = & (\rho'^2 + k^2\cos^2\theta)(d\theta^2 + \sin^2\theta\,d\phi^2) \\ & + 2(d\sigma' - k\sin^2\theta\,d\phi)(d\rho' - k\sin^2\theta\,d\phi) \\ & - \left(1 - \frac{2m\rho'}{\rho'^2 + k^2\cos^2\theta}\right)(d\sigma' - k\sin^2\theta\,d\phi)^2 \end{aligned} \tag{4.12}$$

The energy–momentum tensor (4.9) reduces

$$R_{\mu\nu} = R_{\mu}{}^{1}{}_{\nu} + R_{\mu}{}^{2}{}_{\nu} \tag{4.13}$$

where

$$R_{\mu}{}^{1}{}_{\nu} = -2\dot{m}(z'\bar{z}')^2\rho'^2 k_\mu k_\nu$$

$$k_\mu = \delta^3{}_\mu - \delta^2{}_\mu k\sin^2\theta \tag{4.14a}$$

and

$$R_{12}{}^{2} = -2\rho' k^3 \sin^3\theta \cos\theta \dot{m}(z'\bar{z}')^2 \tag{4.14b}$$

$$R_{13}{}^{2} = 2\rho' k^2 \sin\theta \cos\theta \dot{m}(z'\bar{z}')^2 \tag{4.14c}$$

$$R_{22}{}^{2} = \rho' k^4 \sin^6\theta \ddot{m}(\bar{z}'z')^2 - 4\rho'^4 k^2 \sin^4\theta \dot{m}(z'\bar{z}')^3 - 2\rho'^2 k^4 \sin^4\theta \dot{m}(z'\bar{z}')^3 + 2k^6 \sin^4\theta \cos^2\theta \dot{m}(z'\bar{z}')^3 + 2\rho'^2 k^2 \sin^4\theta (z'\bar{z}')^2 \dot{m} \tag{4.14d}$$

$$R_{23}{}^{2} = -\rho' k^3 \sin^4\theta \ddot{m}(z'\bar{z}')^2 + 4\rho'^2 k^3 \sin^4\theta \dot{m}(z'\bar{z}')^3 - 2k^3 \sin^2\theta \dot{m}(z'\bar{z}')^2 + k\sin^2\theta \dot{m}(z'\bar{z}') \tag{4.14e}$$

$$R_{33}{}^{2} = \rho' k^2 \sin^2\theta \ddot{m}(z'\bar{z}')^2 - 4\rho'^2 k^2 \sin^2\theta \dot{m}(z'\bar{z}')^3 + 2k^2 \sin^2\theta \dot{m}(z'\bar{z}')^2 \tag{4.14f}$$

The part $R^1{}_{\mu\nu}$ represents a radiative field, while $R^2{}_{\mu\nu}$ corresponds to a nonradiative field. (Our expression for the energy–momentum tensor contains some additional terms which are missing in Carmeli and Kaye's paper.)

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